Center for Academic Resources in Engineering (CARE) Peer Exam Review Session

Math 220/221 – Calculus I

Midterm 6 Worksheet Solutions

The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: Apr 24, 4-5:30pm Justin and Bhagya   Session 2: Apr 25, 4-5:30pm Cindy and Stef

Can't make it to a session? Here's our schedule by course:

https://care.engineering.illinois.edu/tutoring-resources/tutoring-schedule-by-course/

Solutions will be available on our website after the last review session that we host, as well as posted in the Zoom chat 30 minutes prior to the end of the session

Step-by-step login for exam review session:

1. Log into Queue @ Illinois
2. Click "New Question"
3. Add your NetID and Name
4. Press "Add to Queue"
5. Join the Zoom link in the staff message

Please do not log into the Zoom call without adding yourself to the queue

Good luck with your exam!

1. Evaluate each of the following limits:

(a)

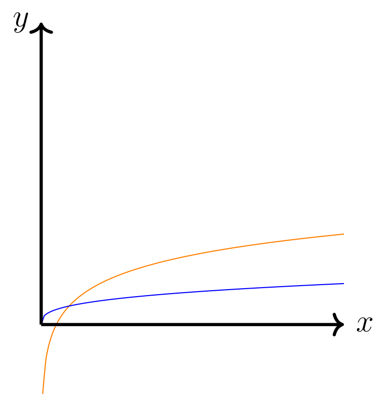
$$\lim_{x\to\infty} \frac{2\ln(x)}{\sqrt[3]{x}}$$

$$\lim_{x\to\infty} \frac{2\ln(x)}{\sqrt[3]{x}} = 2\lim_{x\to\infty} \frac{\ln(x)}{\sqrt[3]{x}}$$

Using l'Hopital:

$$2\lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{3}x^{-2/3}} = 2\lim_{x\to\infty} \frac{3}{x^{1/3}} = 2(0) = 0$$

$$\left(\text{Blue} = \sqrt[3]{x}, \text{ orange} = 2\ln(x)\right)$$



(b)

$$\lim_{x\to 0} \frac{e^{10x}-1}{5x}$$

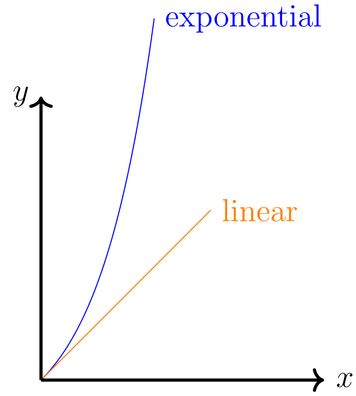
For this limit we must employ l'Hopital's Rule

$$\lim_{x\to 0} \frac{e^{10x}-1}{5x} = \lim_{x\to 0} \frac{10e^{10x}}{5} = 2$$

(c)

$$\lim_{x\to\infty} \frac{e^{10x}-1}{5x}$$

The numerator approaches infinity much more rapidly than the denominator, thus $\lim_{x\to\infty} \to \infty$.



(Note about plots, these plots do not have the coefficients attached to x due to size constraints of the page. The general relation still holds and, in fact, is exacerbated by the coefficients)

2. Check if the Fundamental Theorem of Calculus applies, then evaluate the given integral if it does.

(a)

$$\frac{d}{dx}\int_{-\pi}^{x} \sin(t)dt, \qquad x \in [-\pi, \pi]$$

This takes the form $\frac{d}{dx}\int_a^x f(t)dt$ where f is continuous on the given interval. We therefore can say immediately that this evaluates to $f(x)$, or in this case, $\boxed{\sin(x)}$ on this interval.

Alternatively, one can evaluate the integral as follows:

$$\begin{aligned}\frac{d}{dx}\int_{-\pi}^{x} \sin(t)dt &= \frac{d}{dx}(-\cos(x) - (-\cos(\pi))\\ &= \frac{d}{dx}(-\cos(x) - 1)\\ &= \boxed{\sin(x)}\end{aligned}$$

(b)

$$\frac{d}{dx}\int_{-5}^{x} \frac{1}{t}dt, \qquad x \in [-5, 3]$$

This appears to take a similar form as part (a), but it is important to notice that the given function $1/x$ is *not* continuous on the given interval (it has a discontinuity at $x = 0$). Therefore, the FTC does not apply to this problem. It *would* apply on an interval such as $[-5, -1]$.

3. Evaluate the following indefinite integral:

$$\int \frac{\sin^2(x)}{\sec(x)\csc^4(x)}dx$$

$$\int \frac{\sin^2(x)}{\frac{1}{\cos(x)*} * \frac{1}{\sin^4(x)}}dx = \int \sin^6(x)\cos(x)dx$$

Need to use u-sub for this problem: $(u = \sin(x))$ and $(du = \cos(x)dx)$.

$$\int u^6 du = \frac{1}{7}u^7 + C = \frac{1}{7} * \sin^7(x) + C$$

4. Evaluate the indefinite integral:

$$\int \frac{e^{9x}}{e^{18x}+1}dx$$

Need to use u-sub to solve this problem: $u = e^{9x}$ and

$$\frac{1}{9}du = e^{9x}dx$$

$$\int \frac{1}{u^2+1} * \frac{1}{9}du = \frac{1}{9}\big(\arctan(u)\big) + C = \frac{1}{9}\big(\arctan(e^{9x})\big) + C$$

5. At t hours, a population of bacteria is growing at a rate of

$$r(t) = \frac{21e^{t^{1/2}}}{t^{1/2}} \text{ bacteria per hour}$$

Compute the change in population size between times $t = 169$ s and $t = 225$ s. Simplify your answer.

Net change in population from t = 169 to t = 225 is defined as:

$$\int_{169}^{225} r(t)dt$$

$$= \int_{169}^{225} \frac{(21 * e^{t^{1/2}})}{t^{1/2}} dt$$

$\rightarrow$ u-sub $u = t^{\frac{1}{2}}$ and $2du = (\frac{1}{t^{\frac{1}{2}}})dt$ and $t = 169$ equates to $u = 13$ and $t = 225$ equates to $u = 15$

$$\int_{13}^{15} (21 * 2 * e^{u})du = 42 * e^{15} - 42 * e^{13} \text{ bacteria}$$

6. Express the definite integral as the limit of Riemann Sums. Do not evaluate the limit.

$$\int_{-3}^{5} x^2 e^{\sin(x)} dx$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x \rightarrow \lim_{n\to\infty} \sum_{k=1}^{n} (x_k)^2 (\Delta x)(e^{\sin(x)})$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(-3 + \left(\frac{8k}{n}\right)\right)^2 \left(e^{\sin\left(-3+\frac{8k}{n}\right)}\right)\left(\frac{8}{n}\right)$$

$$f(x) = x^2 * e^{\sin(x)}$$

$$\Delta x = \frac{(b-a)}{n} = \frac{8}{n}$$

$$x = a + k * (\Delta x) = -3 + \left(\frac{8}{n}\right)k$$

NOTE: All parts of the worksheet after part (a) of the next problem may be relevant for the Final Exam, but they are not covered on Exam 6.

7. Let R be the finite region bounded by the graphs of $y = 3\sin(x)$, $y = 6$, and $x = \pi$. Set up, but do not evaluate, definite integrals which represent the following quantities. Integrate with respect to x.

(a) The area of the region, R.

$$A = \int_{x-min}^{x-max} (Y_{top} - Y_{bottom})dx = \int_0^{\pi} (6 - 3\sin(x))dx$$

(b) The volume of the solid formed when R is revolved around the line $y = 8$.

$$V = \int_{x-min}^{x-max} (\text{cross-sectional area})dx = \int_0^{\pi} ((r_{out})^2 - (r_{in})^2)$$

$$V = \int_0^{\pi} ((8 - 3\sin(x))^2 - (8-6)^2)dx$$

(c) The volume of the solid formed when R is revolved around the line $x = -2$.

$$V = \int_{x-min}^{x-max} (\text{surface area})dx = \int_0^{\pi} 2\pi * r * h dx = \int_0^{\pi} 2\pi * (x+2) * (6 - 3\sin(x))dx$$

8. Find the average value of the function below on the interval $[1, 9]$. Simplify.

$$f(x) = \frac{8x}{x^2 + 9}$$

Average Value

$$\left(\frac{1}{9-1}\right) * \int_1^9 \frac{8x}{(x^2+9)} dx$$

Need to use u-sub and set $u = x^2 + 9$ and $4du = 8xdx$

Average-value

$$\frac{1}{8}\int_{10}^{90} \left(\frac{4}{u}\right) du = \frac{1}{2}\big(\ln(90) - \ln(10)\big) = \frac{1}{2}\ln(9)$$

9. Some of the values of a polynomial $f(x)$ are shown below in the table. If $g(x) = 8xf'(x^2)$, then find the average value of $g(x)$ on the interval $[0, 2]$. Simplify your answer.

x	$f(x)$
0	3
1	5
2	8
3	13
4	21
5	34
6	55
7	89
8	144
9	233

Average-value

$$\left(\frac{1}{2-0}\right)\int_0^2 g(x)dx = \frac{1}{2}\int_0^2 8x * f'(x)dx$$

$\rightarrow$ use u-sub where $u = x^2$ and $4du = 8xdx$

Average-value

$$\frac{1}{2}\int_0^4 4f'(u)du = 2\big(f(4) - f(0)\big) = 36$$

10. Use a linear approximation to estimate

$$\ln\left(\frac{95}{100}\right)$$

Write your answer as either a simplified fraction or a decimal value.

$f(x) = ln(x)$ so we need to find the tangent line at x = 1.

$f'(x) = \frac{1}{x}$ at $f'(1) = 1$

So line is: $0 = 1 + b$

$b = -1$ so $L(x) = x - 1$

At x =1, $f(x)$ is approximately equal to L(x) so $L(\frac{95}{100}) = \frac{95}{100} - 1 = \frac{-1}{20} = -0.05$