



## Center for Academic Resources in Engineering (CARE) Peer Exam Review Session

Math 285 — Intro Differential Equations

### Midterm 1 Worksheet Solutions

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*The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.*

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Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: Feb 24, 7-8:30pm Elena and Amy	Session 2: March 1, 7-8:30pm Rehnuma and Chris
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Can't make it to a session? Here's our schedule by course:

<https://care.engineering.illinois.edu/tutoring-resources/tutoring-schedule-by-course/>

Solutions will be available on our website after the last review session that we host, as well as posted in the zoom chat 30 minutes prior to the end of the session

Step-by-step login for exam review session:

1. Log into [Queue @ Illinois](#)
2. Click "New Question"
3. Add your NetID and Name
4. Press "Add to Queue"
5. Join the zoom link in the staff message

**Please do not log into the zoom call without adding yourself to the queue**

Good luck with your exam!

1. Consider the initial value problem:

$$(16 - t^2)y' + t^3y = \cos\left(\frac{t}{2}\right) \quad y(1) = -5$$

On what interval is the unique solution certain to exist?

- A)  $(0, 2\pi)$
- B)  $(-2\pi, 0)$
- C)  $(-4, 0)$
- D)  $\boxed{(-4, 4)}$
- E)  $(-\infty, -4)$

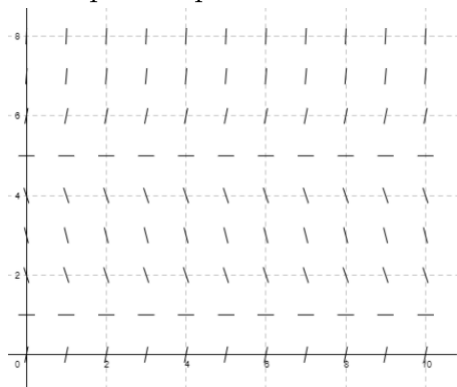
Plug the differential equation in standard form:

$$y' + \frac{t^3}{(4 - t)(4 + t)}y = \cos\left(\frac{t}{2}\right) \frac{1}{(4 - t)(4 + t)}$$

The solution is certain to exist in one of the intervals of continuity of the coefficient functions.

Those intervals are  $(-\infty, -4)$ ,  $(-4, 4)$  and  $(4, \infty)$ . Choose the interval containing the initial  $t$ -value, which is  $t = 1$ . So the answer is  $(-4, 4)$

2. Which equation produces the direction field below?



A)  $\frac{dy}{dx} = (x-1)(x-5)$

B)  $\frac{dy}{dx} = xy$

C)  $\frac{dy}{dx} = (y-1)(y-5)$

D)  $\frac{dy}{dx} = (x-1)(y-5)$

E)  $\frac{dy}{dx} = y^2$

Look at equilibrium solutions to find asymptotes  $y = 1$  and  $y = 5$

3. Which of the following equations are linear?

(I)  $\frac{d^2y}{dt^2} + e^y = 6t + 5$

(II)  $(2t^3 + 6)\frac{d^5y}{dt^5} - \frac{d^3y}{dt^3} + 4y = t \cos(t-1)$

(III)  $u_y = uu_{xx} - u_{xy}$

(IV)  $u_{xx} + xu_{xt} + t^2u_{tt} = \sin(x+2t)$

A) (I), (II), (III)

B) (II) and (IV)

C) (I) and (IV)

D) (II)

E) (IV)

Linear DEs cannot have products, or other nonlinear functions, of the dependent variable or its derivatives.

4. What's the order of the following differential equation?

$$\cot(y)y''' + (t^2 + t + 9)y' - \ln(xy^2)y + 6y^9 = \sin(3t^5 + 1)$$

- A) 9  
 B) 1  
 C) ☒ 3  
 D) 5  
 E) None; not a linear ODE

Highest derivative determines order

5. Consider the functions:

$$f(x, t) = e^{-4t} \sin(x)$$

$$g(x, t) = e^{-t} \cos(2x)$$

$$h(x, t) = \sin(x) \cos(t)$$

Which of these functions are solutions of the partial differential equation  $u_t = 4u_{xx}$

- A)  $f$  and  $g$   
 B)  $f$  and  $h$   
 C)  $g$  only  
 D)  $g$  and  $h$   
 E) ☒  $f$  only

Plug into expression to determine if the condition is satisfied:

$$\begin{aligned} f(x, t) \\ \frac{df}{dt} &= 4 \frac{d^2 f}{dx^2} \\ -4e^{-4t} \sin(x) &= 4(-\sin(x)) \end{aligned}$$

$$g(x, t)$$

$$\frac{dg}{dt} = 4 \frac{d^2 g}{dx^2}$$

$$\begin{aligned} h(x, t) \\ \frac{dh}{dt} &= 4 \frac{d^2 h}{dx^2} \\ -\sin(x) \sin(t) &\neq -4 \sin(x) \cos(t) \end{aligned}$$

$$-e^{-t} \cos(2x) \neq 4(-4e^{-t} \cos(2x))$$

6. Consider the population model.  $y' = -r(1 - \frac{y}{T})y$   
Find and define the equilibrium solutions

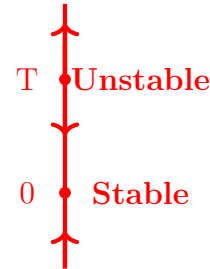
- A)  $y(t) = 0$  is asymptotically stable;  $y(t) = T$  is unstable  
 B)  $y(t) = r$  is asymptotically unstable;  $y(t) = T$  is stable  
 C)  $y(t) = r$  is asymptotically stable;  $y(t) = 0$  is unstable  
 D)  $y(t) = 0$  is asymptotically unstable;  $y(t) = T$  is stable  
 E) None of the above

Draw a phase line

Plug in  $y > T$ , we get  $y' > 0$

Plug in  $T > y > 0$ , we get  $y' < 0$

Plug in  $y < 0$ , we get  $y' > 0$



7. What's the solution to  $\frac{dz}{dx} = -4z - 3$

A)  $z = Ce^{-4x} + \frac{3}{4}$

B)  $z = Ce^{4x} + \frac{3}{4}$

C)  $z = Ce^{-4x} - \frac{3}{4}$

D)  $z = Ce^{-3x} - \frac{4}{3}$

E)  $z = Ce^{-4x} - \frac{4}{3}$

The differential equation is 1st order constant coefficient linear of the form

$$\frac{dz}{dx} = az - b$$

with  $a = -4$  and  $b = 3$  The solution is

$$z = \frac{b}{a} + Ce^x = Ce^{-4x} - \frac{3}{4}$$

(Alternatively, an integrating factor of  $e^{\int -4dx}$  can be used to solve the problem)

8. Solve the following differential equation

$$\frac{dy}{dx} - \frac{2y}{7} = \frac{3e^{5x}}{y^6}$$

(Hint: Bernoulli)

This is a Bernoulli equation, with  $n = -6$

Make a substitution  $v = y^7$

$$\frac{dv}{dx} = 7y^6 \frac{dy}{dx}$$

Multiplying both sides of the equation by  $y^6$ , we obtain

$$y^6 \frac{dy}{dx} - \frac{2}{7} y^7 = 3e^{5x}$$

This gives us

$$\begin{aligned} \frac{1}{7}(v' - 2v) &= 3e^{5x} \\ v' - 2v &= 21e^{5x} \end{aligned}$$

Multiplying by the integrating factor  $\mu(x) = e^{\int -2dx} = e^{-2x}$ , we obtain

$$[e^{-2x}v]' = 21e^{3x}$$

Integrate both sides

$$\begin{aligned} e^{-2x}v &= C + 7e^{3x} \\ v &= Ce^{2x} + 7e^{5x} \end{aligned}$$

Solving for  $y$  we obtain:

$$y(x) = (Ce^{2x} + 7e^{5x})^{\frac{1}{7}}$$

9. Solve  $x\left(\frac{dy}{dx}\right) = 2y + x$

We rewrite the equation in the following form:

$$\frac{dy}{dx} - 2\frac{y}{x} = 1$$

Note that it is a 1st order linear equation

Computing the integrating factor:

$$\mu(x) = e^{-\int \frac{2}{x} dx} = e^{-2\ln(x)} = x^{-2}$$

Multiplying both sides by this integrating factor, we get  $[x^{-2}y]' = x^{-2}$

Now we integrate both sides

$$\int [x^{-2}y]' dx = \int x^{-2} dx$$

We get  $x^{-2}y = -x^{-1} + C$

$$\boxed{y(x) = -x + Cx^2}$$

10. Consider  $y'' + 5y' + 6y = 0$  with  $y(0) = 1$  and  $y'(0) = -2$ . Find the fundamental set of solutions (and prove that it is). (Hint: Prove with Wronskian)

Write the characteristic equation

$$r^2 + 5r + 6 = (r + 2)(r + 3)$$

Therefore we get  $r = -2, -3$

$$\text{Thus } y_1 = e^{-2t}; y_2 = e^{-3t}$$

Now we compute the Wronskian to determine if they're a fundamental set (linearly independent)

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{vmatrix} = e^{-2t}(-3e^{-3t}) - e^{-3t}(-2e^{-2t}) = -e^{-5t}$$

$$-e^{-5t} \neq 0$$

Therefore  $y(t) = C_1 e^{-3t} + C_2 e^{-2t}$  is the general solution and  $e^{-3t}, e^{-2t}$  is the fundamental set