Center for Academic Resources in Engineering (CARE) Peer Exam Review Session

Math 241 – Calculus III

Midterm 2 Worksheet Solutions

The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: April 11, 4-5:30pm Pieter and David Session 2: April 12, 7:30-9pm Nidhi and Danny

Can't make it to a session? Here's our schedule by course:

`https://care.engineering.illinois.edu/tutoring-resources/tutoring-schedule-by-course/`

Solutions will be available on our website after the last review session that we host, as well as posted in the zoom chat 30 minutes prior to the end of the session

Step-by-step login for exam review session:

1. Log into Queue @ Illinois
2. Click "New Question"
3. Add your NetID and Name
4. Press "Add to Queue"
5. Join the zoom link in the staff message

Please do not log into the zoom call without adding yourself to the queue

Good luck with your exam!

1. Calculate the following derivatives:

(a) Find $\frac{\mathrm{d}f}{\mathrm{d}t}$ for $f(x,y) = xe^{xy}$, $x(t) = t^2$, $y(t) = \frac{1}{t}$

Applying the chain rule:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

$$\frac{\partial f}{\partial x} = e^{xy} + x(ye^{xy}) \qquad \frac{dx}{dt} = 2t$$

$$\frac{\partial f}{\partial y} = x^2 e^{xy} \qquad \frac{dy}{dt} = -\frac{1}{t^2}$$

$$\frac{df}{dt} = \left[e^{xy} + x(ye^{xy})\right](2t) - [x^2 e^{xy}]\left(-\frac{1}{t^2}\right)$$

$$\boxed{\frac{df}{dt} = 2te^t + t^2 e^t}$$

(b) Find f_t for $f(x,y) = 2xy$, $x(s,t) = st$, $y(s,t) = s^2t^2$

Applying Chain Rule:

$$f_t = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{\partial f}{\partial x} = 2y,\ \frac{\partial x}{\partial s} = s,\ \frac{\partial f}{\partial y} = 2x,\ \frac{\partial y}{\partial s} = 2ts^2$$

$$\boxed{f_t = 6s^3t^2}$$

2. The vector field $\vec{F} = \langle 2xy + 2x + y^2, 2xy + 2y + x^2 \rangle$ is conservative. Find a potential function f for $\vec{F}$ (a function with $\nabla f = \vec{F}$)

$$\vec{F} = \langle f_x, f_y \rangle = \langle 2xy + 2x + y^2, 2xy + 2y + x^2 \rangle$$
$$\int f_x = yx^2 + x^2 + xy^2 + C(y)$$
$$\int f_y = xy^2 + y^2 + yx^2 + C(x)$$

Looking at all the terms and comparing with $\vec{F}$, we know that $C(y) = y^2$ and $C(x) = x^2$, therefore the potential function is:

$$\boxed{f(x, y, z) = yx^2 + x^2 + xy^2 + y^2}$$

3. A tiny spaceship is orbiting a path given by $x^2 + y^2 = 4$. The solar radiation at a point (x, y) in the plane of the orbit is $f(x, y) = xy + 2y$.

   Use the method of *Lagrange multipliers* to find the maximum value and minimum value of solar radiation experienced by the tiny spaceship in its orbit.

   The function to be maximized/minimized is f, subject to the constraint $x^2 + y^2 = 4$, which we will call g.

$$\nabla f = \langle y, x + 2 \rangle \qquad \nabla g = \langle 2x, 2y \rangle$$

   We have 3 equations to work with. We know

$$\nabla f = \lambda \nabla g$$

   gives the following system of equations

$$y = \lambda(2x)$$
$$x + 2 = \lambda(2y)$$
$$x^2 + y^2 = 4$$

   Where the last equation is the constraint. We solve for λ in the first two equations:

$$\frac{y}{2x} = \lambda \text{ and } \frac{x+2}{2y} = \lambda$$

Set them equal to each other to get an equation with just x and y:

$$\frac{y}{2x} = \frac{x+2}{2y}$$
$$2y^2 = 2x^2 + 4x$$
$$y^2 = x^2 + 2x$$

Plug this in to the constraint

$$x^2 + y^2 = 4$$
$$x^2 + x^2 + 2x = 4$$
$$\text{We get } x = 1,\ -2$$

Obtain the corresponding values for y:
For $x = 1$

$$y^2 = x^2 + 2x \qquad y = -\sqrt{3},\ \sqrt{3}$$

For $x = -2$

$$y^2 = x^2 + 2x \qquad y = 0$$

Now we must test whether these are minimum or maximum points. We have three points to test in our radiation function $f(x, y) = xy + 2y$

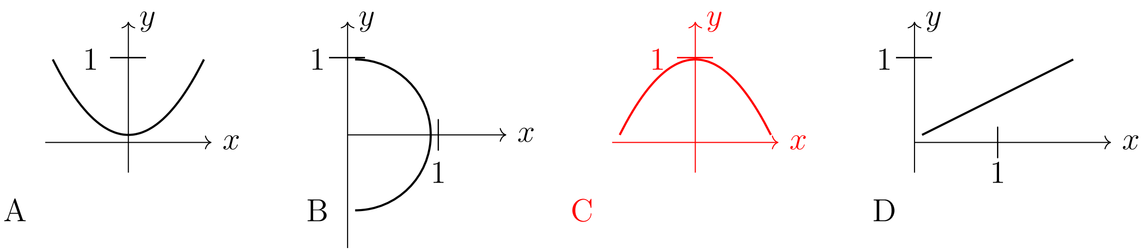
$$f(1, \sqrt{3}) = 3\sqrt{3}$$
$$f(1, -\sqrt{3}) = -3\sqrt{3}$$
$$f(-2, 0) = 0$$
$$\boxed{\text{Max: } 3\sqrt{3}}$$
$$\boxed{\text{Min: } -3\sqrt{3}}$$

4. Let $r(t) = \langle \sin(t), \cos^2 t \rangle, 0 \leq t \leq 2\pi$. Which graph below represents this curve?

Converting this into a Cartesian equation gives

$$y + x^2 = 1 \to y = 1 - x^2$$

Which is a concave down parabola with its vertex at $(0, 1)$



5. Let $f(x, y)$ be a differentiable function on the disk $\{D : x^2 + y^2 \leq 400\}$, where:

(I) $f(x, y) = 19$ for every point on the boundary of the disk $x^2 + y^2 = 400$

(II) $f(0, 0) = 7$

(III) $f(x, y)$ has only one critical point which is at $(-1, 2)$

Decide which statement is true:

A) $f(-1, 2) > 7$

B) $\boxed{f(-1, 2) < 7}$

C) $f(-1, 2) = 7$

D) Not enough information is given

From the extreme value theorem we know that the maximum and minimum of the function in the domain D must (if they exist) be at the critical points or some points on the boundary.

Suppose $f(-1, 2) \geq 7$, then the minimum of the function is not at the point $(-1, 2)$ since $f(0, 0)$ has a smaller value. This would imply another critical point below $f(-1, 2)$.

The minimum is also not on the boundary since $f(x, y) = 19 > 7$ for every point on the boundary of the disk.

So the only way to satisfy the condition that there is one critical point is make $f(-1, 2) < 7$. This means it is below the origin.

6. Consider the function $f(x, y) = x^3 + y^3 + 3xy$

(a) The critical points of f are $(0, 0)$ and $(-1, -1)$. Classify them into local minima, local maxima and/or saddle points

$$f_x = 3x^2 + 3y \qquad f_y = 3y^2 + 3x$$

$$f_{xx} = 6x \qquad f_{yy} = 6y$$

$$f_{xy} = f_{yx} = 3$$

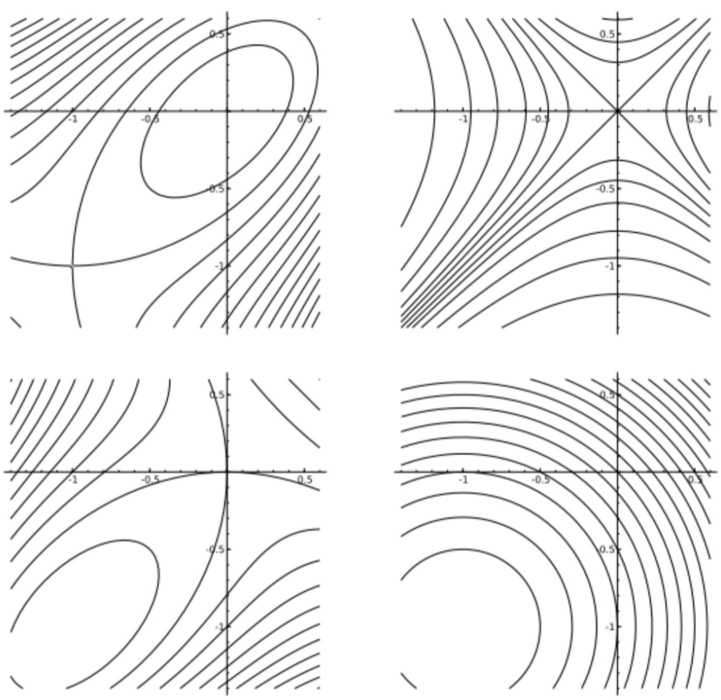
$$\text{At } (0,0)\ D = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} = -9 < 0 \rightarrow \text{SADDLE}$$

$$\text{At } (-1,-1)\ D = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} -6 & 3 \\ 3 & -6 \end{bmatrix} = 36 - 9 = 27$$

$$27 > 0 \text{ and } f_{xx} = -6 < 0 \rightarrow \text{LOCAL MAX}$$

(b) Based on your answer in (a), identify the correct contour diagram of f

Bottom left is correct



7. What is the partial derivative of $f(x, y, z) = e^x \sin(yz) z^3 \ln(y)$ with respect to x.

   It's the same function.