ILLINOIS

Center for Academic Resources in Engineering (CARE)
Peer Exam Review Session

Math 231 – Calculus II

Midterm 4 Worksheet Solutions

The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: Mar 21, 4-5:30pm Alberto and Kristin; Session 2: Mar 22, 8:30-10pm Steven and Ayush

Can't make it to a session? Here's our schedule by course:

https://care.engineering.illinois.edu/tutoring-resources/tutoring-schedule-by-course/

Solutions will be available on our website after the last review session that we host, as well as posted in the Zoom chat 30 minutes prior to the end of the session.

Step-by-step login for exam review session:

1. Log into Queue @ Illinois
2. Click "New Question"
3. Add your NetID and Name
4. Press "Add to Queue"
5. Join the Zoom link in the staff message

Please do not log into the Zoom call without adding yourself to the queue.

Good luck with your exam!

1. Determine whether the series converges or diverges:

$$\sum_{n=5}^{\infty} \frac{1}{n(\ln(n))^2}$$

Use the integral test.

$= \int_5^\infty \frac{1}{x(\ln(x))^2} dx$ $[u = \ln(x),\ du = \frac{1}{x}]$

$= \frac{-1}{\ln(x)}|_5^\infty = \frac{1}{\ln(5)} \rightarrow$ so the integral converges. Both converge.

2. Determine whether the series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{n^2}{n^2+1}$$

Use the divergence test:

$\lim_{n\to\infty} \frac{(n^2)}{n^2+1} = 1 \rightarrow$ series diverges.

3. The sum $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, yet has no known closed-form expression (i.e. We don't know the exact value of the sum). Use the Integral Test to find the smallest and largest possible values for the sum.

The integral test is valid here, since $f(n) = \frac{1}{n^3}$ is a positive, decreasing function that goes to 0. We may conclude that

$$\int_1^\infty \frac{1}{x^3}\, dx \le \sum_{n=1}^{\infty} \frac{1}{n^3} \le \frac{1}{1^3} + \int_1^\infty \frac{1}{x^3}\, dx$$
$$\frac{1}{2} \le \sum_{n=1}^{\infty} \frac{1}{n^3} \le \frac{3}{2}$$

In fact, $\sum_{n=1}^{\infty} \frac{1}{n^3} \approx 1.2020$.

4. Determine whether the series converges or diverges:

$$\sum_{n=3}^{\infty} \frac{1}{n\ln(n)}$$

Use the integral test:

$f(x) = \frac{1}{x\ln(x)}$ is positive and decreasing and continuous between 3 and ∞.

$\int_3^\infty \frac{1}{x\ln(x)} = \ln(\ln(x))|_3^\infty = \ln(\ln(\infty)) - \ln(\ln(3)) = \infty$

The integral diverges, so the series diverges as well.

5. Find the values of x for which the following series converges, and then find the sum of the series for those values:

$$\sum_{n=0}^{\infty} \frac{\cos^n(x)}{3^n}$$

Identify the geometric series:

$\sum_{n=1}^{\infty} ar^{n-1}$ where $r = \frac{(n+1)-term}{(n)-term} = [\frac{\cos^{n+1}(x)}{3^{n+1}}][\frac{3^n}{\cos^n(x)}]$

$= \frac{\cos^1(x)}{3} = \frac{1}{3}\cos(x)$, which is always < 1, so the series converges for all x.

Converges to $\frac{a}{1-r} \rightarrow$ a = 1 and $r = \frac{\cos(x)}{3} \rightarrow \frac{1}{1-\frac{\cos(x)}{3}} = \frac{3}{3-\cos(x)}$

6. Determine whether the series converges. If it converges, find what it converges to.

$$\sum_{n=1}^{\infty} \frac{1}{(\frac{1}{2})^n}$$

$r = \frac{(n+1)-term}{(n)-term} = [\frac{1}{(\frac{1}{2})^{n+1}}][\frac{(\frac{1}{2})^n}{1}] = \frac{1}{\frac{1}{2}} = 2 \rightarrow$ diverges.

7. Determine whether the series converges. If it converges, find what it converges to.

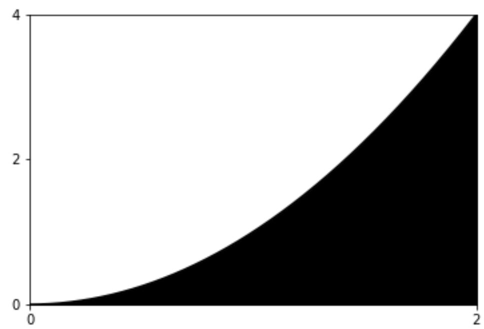
$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

$r = \frac{(n+1)-term}{(n)-term}$

$r = \frac{(n+1)-term}{(n)-term} = [\frac{1}{2^{n+1}}][\frac{2^n}{1}] = \frac{1}{2} \rightarrow$ converges.

$\frac{a}{1-r} = \frac{(\frac{1}{2^1})}{1-\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$

8. Find the M_x, M_y, and the centroid of $y = x^2$ with density λ on $x \in [0, 2]$.



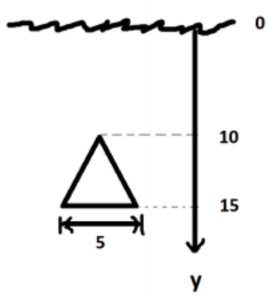
$mass = \lambda \int_0^2 x^2 dx = \frac{\lambda}{3}x^3|_0^2 = \frac{8\lambda}{3}$

$M_x = \frac{\lambda}{2}\int_0^2 (x^2)^2 dx = \frac{\lambda}{10}x^5|_0^2 = \frac{16\lambda}{5}$

$M_y = \lambda \int_0^2 x(x^2) dx = \frac{\lambda}{4}x^4|_0^2 = 4\lambda$

Centroid (x,y) $= (\frac{M_y}{mass}, \frac{M_x}{mass}) = (\frac{3}{2}, \frac{6}{5})$

9. Determine the hydrostatic force on the triangle given the density of water $\rho = 1000\text{kg/m}^3$ with a depth y and $g = 9.8\text{m/s}^2$.

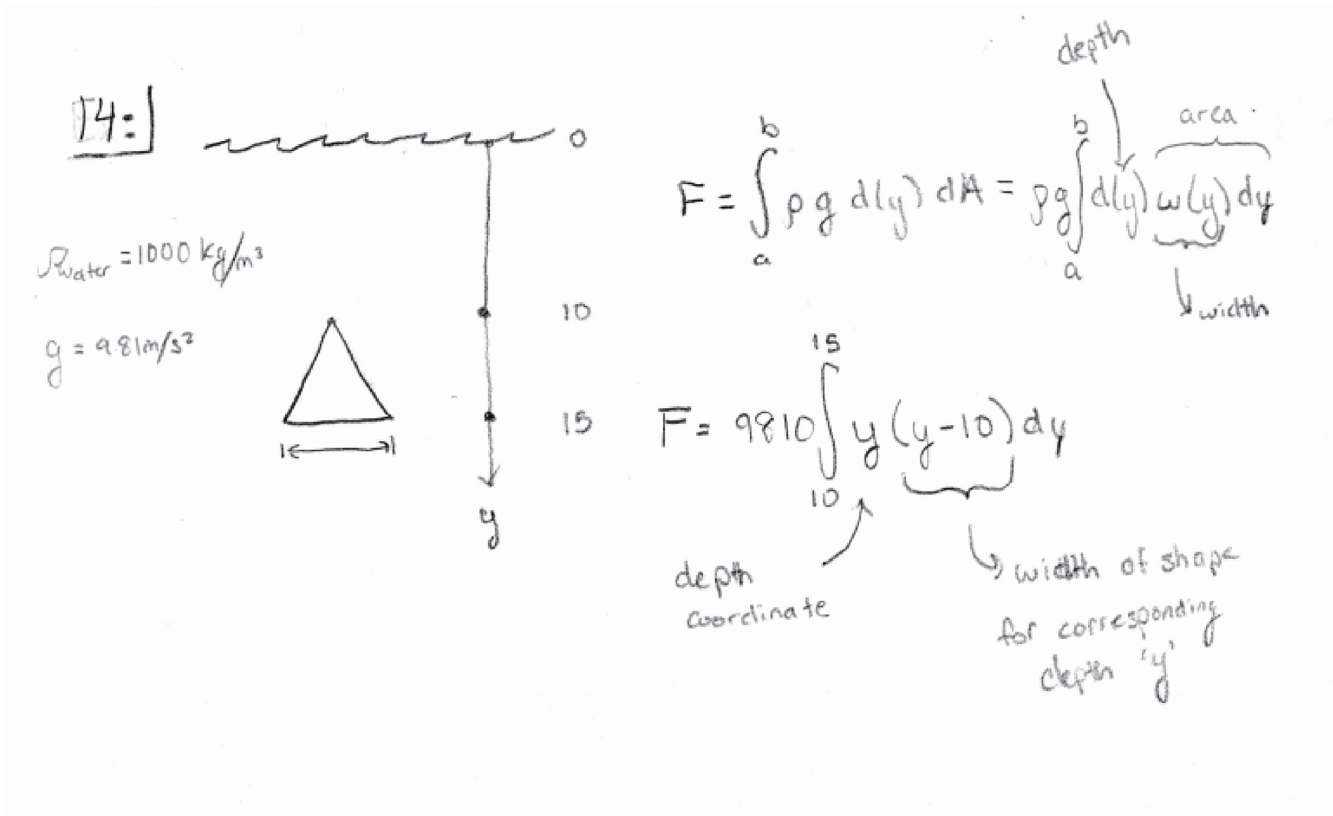


See image below for what each term means within the integral

$F = \int_a^b \rho g d(y) dA = \rho g \int_a^b d(y)w(y)dy$

$F = 9810 \int_{10}^{15} y(y-10)dy$

$F = 1635000$ N



10. Consider the curve y = 5ln(x) between the points (1,0) and (e,5).

A. SET UP, BUT DO NOT EVALUATE, a dx-integral which represents the arc length of the curve.
B. SET UP, BUT DO NOT EVALUATE, a dy-intergral which represents the arc length of the curve.
C. SET UP, BUT DO NOT EVALUATE, a definite integral which represents the surface area of the surface obtained by rotating the curve around the line y =10.

(a) $ds = \sqrt{1 + (\frac{dy}{dx})^2}dx$, $y' = \frac{5}{x}$, $ds = \int_1^e \sqrt{1 + (\frac{5}{x})^2}$

(b) $y = 5\ln(x)$, $e^{\frac{y}{5}} = e^{\ln(x)}$, $x = e^{\frac{y}{5}}$, $x' = \frac{e^{\frac{y}{5}}}{5}$, $ds = \int_0^5 \sqrt{1 + (\frac{e^{\frac{y}{5}}}{5})^2}dy$

(c) $SA = \int 2\pi y ds$, $ds = \sqrt{1 + (\frac{e^{\frac{y}{5}}}{5})^2}dy$, $SA = 2\pi \int_0^5 (10 - y)\sqrt{1 + (\frac{e^{\frac{y}{5}}}{5})^2}dy$

11. The profile $y = \sqrt{4-x^2}$ on the interval $x \in [-1, 1]$ is revolved around the x-axis. Find the surface area of this surface.

Use the following formula to find the surface area of an arc rotated about the x-axis:

$$S = \int 2\pi y ds$$

$$\text{where } ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

We choose to integrate with respect to x since we are given that interval.

$$S = \int_{-1}^{1} 2\pi y \sqrt{1 + \left(\frac{-x}{\sqrt{4-x^2}}\right)^2} dx$$

We can simplify the square root term from the ds.

$$\sqrt{1 + \left(\frac{-x}{\sqrt{4-x^2}}\right)^2} = \sqrt{1 + \frac{x^2}{4-x^2}} = \sqrt{\frac{4-x^2}{4-x^2} + \frac{x^2}{4-x^2}} = \sqrt{\frac{4}{4-x^2}} = \frac{2}{\sqrt{4-x^2}}$$

$$S = \int_{-1}^{1} 2\pi y \frac{2}{\sqrt{4-x^2}} dx$$

We can then write y in terms of x and simplify.

$$S = \int_{-1}^{1} 2\pi \sqrt{4-x^2} \frac{2}{\sqrt{4-x^2}} dx$$

$$S = \int_{-1}^{1} 4\pi dx = 8\pi$$

12. Compute the arc length of the function $y = 1 + 2x^{\frac{3}{2}}$ between $x = 0$ and $x = 1$

(a) $\frac{14}{9}$

(b) $\frac{10}{9}$

(c) $\frac{2}{9}\sqrt{10}$

(d) $\frac{2}{27}(10\sqrt{10} + 1)$

(e) $\boxed{\text{None of the above}}$

$$ds = \sqrt{1 + (\frac{dy}{dx})^2}$$

$$y = 1 + 2x^{\frac{3}{2}}, \frac{dy}{dx} = 3x^{\frac{1}{2}}$$

$$S = \int_0^1 \sqrt{1 + (3x^{\frac{1}{2}})^2} dx$$

$$S = \int_0^1 \sqrt{1 + 9x} dx$$

$u = 1 + 9x$, $du = 9dx$, $dx = \frac{1}{9}du$

The u-bounds are: $u = 1 + 9(1) = 10$, $u = 1 + 9(0) = 1$

$$S = \sqrt{u}\frac{1}{9}du$$

$$S = \frac{1}{9}(\frac{2}{3})u^{\frac{3}{2}}|_1^{10}$$

$$(\frac{2}{27})(10^{\frac{3}{2}} - 1^{\frac{3}{2}}$$

$$S = \left(\frac{2}{27}\right)(10\sqrt{10} - 1)$$