



Center for Academic Resources in Engineering (CARE) Peer Exam Review Session

Math 285 — Intro Differential Equations

Midterm 2 Worksheet Solutions

The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: 3/28, 4:30-6pm Gabby and Jesse Session 2: 3/29, 6:30-8pm Rehnuma and Amy

Can't make it to a session? Here's our schedule by course:

<https://care.engineering.illinois.edu/tutoring-resources/tutoring-schedule-by-course/>

Solutions will be available on our website after the last review session that we host, as well as posted in the zoom chat 30 minutes prior to the end of the session

Step-by-step login for exam review session:

1. Log into [Queue @ Illinois](#)
2. Click "New Question"
3. Add your NetID and Name
4. Press "Add to Queue"
5. Join the zoom link in the staff message

Please do not log into the zoom call without adding yourself to the queue

Good luck with your exam!

1. Rewrite the function $y(t) = 4\cos(7t) - 4\sin(7t)$ in phase amplitude.

- A) $\boxed{4\sqrt{2}\cos(7t + \frac{\pi}{4})}$
 B) $4\sqrt{2}\cos(7t - \frac{\pi}{4})$
 C) $4\cos(7t + \frac{\pi}{4})$
 D) $4\sqrt{2}\cos(7t)$

The phase-amplitude form of function

$$A\cos(\omega t) + B\sin(\omega t) = R\cos(\omega t - \delta)$$

$$R = \sqrt{A^2 + B^2} \quad \cos(\delta) = \frac{A}{R} \quad \sin(\delta) = \frac{B}{R}$$

In our case $\omega = 7$, $R = \sqrt{4^2 + (-4)^2} = 4\sqrt{2}$ and $\tan(\delta) = \frac{-4}{4}$ so $\delta = \frac{-\pi}{4}$

2. The differential equation $mu'' + 4u' + 8u = 0$ describes a mass-spring system. For what values of m is the system underdamped?

- A) $m < \frac{1}{2}$
 B) $m < 2$
 C) $\boxed{m > \frac{1}{2}}$
 D) $m > 2$

The system is underdamped when its characteristic equation $mr^2 + 4r + 8 = 0$ has two complex roots

This happens when the discriminant $4^2 - 4m * 8 = 16 - 32m$ is negative

So, $16 - 32m < 0$ thus $m > \frac{1}{2}$

3. For what forcing frequency ω is it possible for the solution to $18y'' + 2y = 81\cos(\omega t)$ to grow without bound?

A) $\omega = 3$

B) $\omega = 9$

C) $\omega = \frac{1}{3}$

D) $\omega = \frac{1}{9}$

This is possible when the system is in resonance, meaning the external frequency ω equals the resonance frequency

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{2}{18}} = \frac{1}{3}$$

4. Consider the following statements

(i) If Y_1 is a particular solution to the DE

$$y'' + p(t)y' + q(t)y = g(t)$$

then $Y_2 = cY_1$ is also a solution to the differential equation, where c is a constant

(ii) If y_1 and y_2 are solutions to

$$ay'' + by' + cy = g(t)$$

then $y_1 - y_2$ is a solution to the homogeneous differential equation $ay'' + by' + cy = 0$ where a , b , and c are constants.

(iii) The solution to

$$ay'' + by' + cy = 0$$

with initial conditions $y(0) = A$ and $y'(0) = B$, is unique on $t \in (-\infty, \infty)$ where a , b , and c are constants.

(iv) Two solutions to a linear second order ODE can cross each other.

Which of the above statements are always true?

A) (i), (iii) and (iv)

B) (iii) and (iv)

C) (ii) and (iii)

D) (ii), (iii) and (iv)

(i) This statement is False:

$$Y_2'' + p(t)Y_2' + q(t)Y_2 = g(t)$$

$$c(Y_1'' + p(t)Y_1' + q(t)Y_1) = cg(t)$$

$cg(t) \neq g(t)$ if $c \neq 1$ not ALWAYS true

(ii) This statement is True

$$a(y_1 - y_2)'' + b(y_1 - y_2)' + c(y_1 - y_2) \\ (ay_1'' + by_1' + cy_1) - (ay_2'' + by_2' + cy_2) = g(t) - g(t) = 0$$

(iii) This statement is True:

The existence and uniqueness theorem states that constant functions are continuous. The initial conditions also exist on the interval.

(iv) This statement is True:

For example, $\sin(t)$ and $\cos(t)$ solve the differential equation $y'' + y = 0$ and their plots cross each other.

5. Consider the following initial value problem $(t - 5)y'' + \csc(t)y' + y = e^t$, $y'(4) = 1$ and $y(4) = \pi$. What is the largest interval on which the initial value problem is guaranteed to exist?

- A) $(0, 5)$
- B) $(0, 2\pi)$
- C) $\boxed{(\pi, 5)}$
- D) $(\pi, 2\pi)$

Rewrite the equation as

$$y'' + \frac{\csc(t)}{(t - 5)}y' + \frac{1}{(t - 5)}y = \frac{e^t}{(t - 5)}$$

The coefficients exist everywhere except $t = 5$ and $t = 0, \pi, 2\pi \dots$

We need to find the interval that contains the point 4 and none of the points that don't exist, which is (C)

6. Identify the form of the particular solution for $y'' - 16y = (t - 3)e^{-4t} + (4t + 3)\sin(2t)$

A) $(At + B)te^{-4t} + (Ct + D)t\sin(2t) + (Et + F)t\cos(2t)$

B) $(At + B)e^{-4t} + (Ct + D)\sin(2t) + (Et + F)\cos(2t)$

C) $(At + B)e^{-4t} + (Ct + D)t\sin(2t) + (Et + F)t\cos(2t)$

D) $(At + B)te^{-4t} + (Ct + D)\sin(2t) + (Et + F)\cos(2t)$

The particular solution Y should be of the same form as the right side of the equation

We might need to multiply by a power of t to avoid duplications with the complementary solution

That is, the terms of Y should not be solutions to the homogeneous DE

$$y_c = C_1e^{4t} + C_2e^{-4t}$$

The term $(t - 3)e^{-4t}$ corresponds to the particular solution term (we multiply by t to avoid these duplicate terms)

$$(At + B)te^{-4t}$$

While the term $(4t + 3)\sin(2t)$ corresponds to the particular solution term

$$(Ct + D)\sin(2t) + (Et + F)\cos(2t)$$

Combine these two solutions to get the form of the solution

7. Find the general solution for the differential equation $y'' + 4y' + 4y = 0$

First find the solution to the homogeneous equation

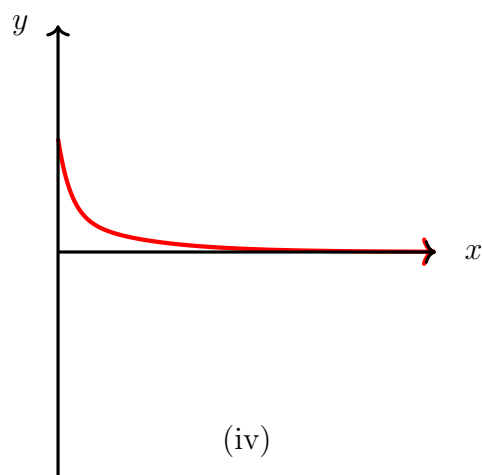
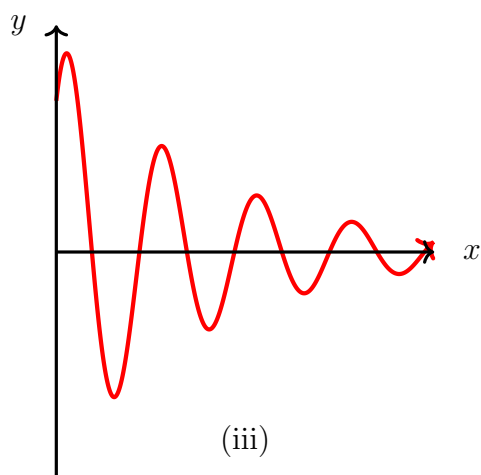
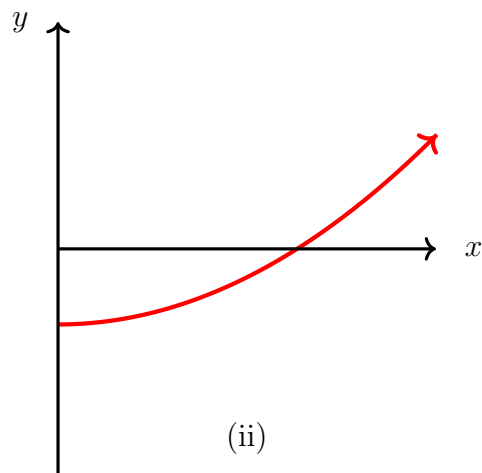
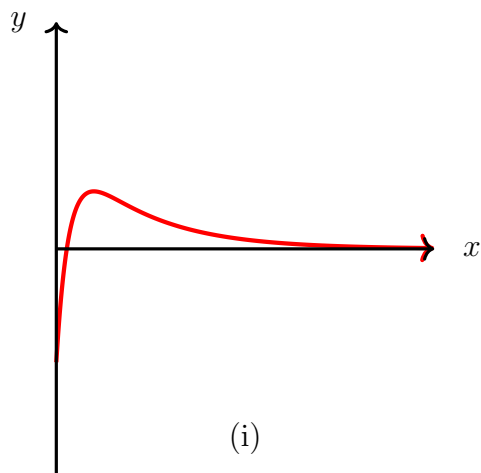
$$r^2 + 4r + 4 = 0$$

This has a repeated root of $r = -2$

Then, we multiply one of the solutions by t to obtain

$$y = C_1e^{-2t} + C_2te^{-2t}$$

8. Which of the following plots represents a solution to the ODE $y'' + 7y' + 6y = 0$?



- A) (ii) only
 B) (iii) only
 C) (i) and (iv) only
 D) (ii) and (iii) only

The characteristic equation

$$r^2 + 7r + 6 = 0$$

has roots $r = -1$ and $r = -6$

The solution is

$$y = c_1 e^{-t} + c_2 e^{-6t}$$

The solution tends to 0 and it equals 0 at most once

9. We know that $y(t) = e^t$ is a solution to the ODE $2ty'' + (1 - 4t)y' + (2t - 1)y = 0$ for $y > 0$. Find the general solution.

We use reduction of order. Re-write the DE in the standard form, as

$$y'' + \frac{1 - 4t}{2t}y' + \frac{2t - 1}{2t}y = 0$$

We look for the general solution in the form $y = vy_1$, where $y_1(t) = e^t$ and v satisfies

$$y_1v'' + (2y_1' + py_1)v' = 0$$

In our case, $p(t) = \frac{1}{2t} - 2$, so we get the equation

$$e^tv'' + \left(2e^t + \left(\frac{1}{2t} - 2\right)\right)e^tv' = 0$$

Simplifying, we get $v'' + 12tv' = 0$

This is the first order ODE for v' ; we can solve it by separating variables (solution shown below), or via integrating factors

$$\text{Write } \frac{dv'}{dt} = \frac{-1}{2t}v', \text{ so } \frac{dv'}{v'} = \frac{-1}{2t}dt$$

$$\text{Integrate } \ln(v') = C - \frac{1}{2}\ln(t)$$

$$v' = \frac{dv}{dt} = Kt^{-\frac{1}{2}}$$

Integrate once more to get $v = A\sqrt{t} + B$, where $A = -2K$

$$\boxed{y(t) = (A\sqrt{t} + B)e^t}$$