ILLINOIS

Center for Academic Resources in Engineering (CARE)
Peer Exam Review Session

Math 285 − Intro Differential Equations

Midterm 3 Worksheet Solutions

The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: Apr 25, 6-7:30pm Gabby and Amy   Session 2: Apr 26, 6:30-8pm Chris and Elena

Can't make it to a session? Here's our schedule by course:

Solutions will be available on our website after the last review session that we host, as well as posted in the zoom chat 30 minutes prior to the end of the session

Step-by-step login for exam review session:

1. Log into Queue @ Illinois

2. Click "New Question"

3. Add your NetID and Name

4. Press "Add to Queue"

5. Join the zoom link in the staff message

   Please do not log into the zoom call without adding yourself to the queue

Good luck with your exam!

Multiple Choice Questions

1. Consider the initial value problem

$$(\cos t)y''' + (t-1)y'' - \frac{1}{t+1}y = t^3 - 1 \qquad y'''(0) = 5, y'(0) = 0, y(0) = \pi$$

On what interval is this problem guaranteed to have a unique solution?

A) $\left(\frac{\pi}{2}, 1\right)$

B) $\boxed{\left(-1, \frac{\pi}{2}\right)}$

C) $(-1, 1)$

D) $(-\infty, \infty)$

Rewrite the DE in its standard form:

$$y''' + \frac{t-1}{\cos t}y'' - \frac{t}{(t+1)\cos t}y = \frac{t^3-1}{\cos t}$$

The functions involved are continuous on $(-1, \frac{\pi}{2})$ (because the denominators are nonzero on that interval)

2. Find the steady-state solution of the heat equation $4u_{xx} = u_t$, $u_x(0,t) = \frac{1}{2}u(0,t)$, $u(3,t) = 15$

A) DNE

B) 15

C) $\boxed{6 + 3x}$

D) $15 - 4e^{-t}$

A steady state solution is of the form

$$v(x) = C_1 + C_2 x$$

We have $v'(0) = \frac{1}{2}v(0)$ so that $C_2 = \frac{1}{2}C_1$ and also $v(3) = 15$ so the $C_1 + 3C_2 = 15$
Solve to obtain $C_1 = 6$ and $C_2 = 3$, giving $v(x) = 6 + 3x$

3. Which of the following sets of functions are linearly INDEPENDENT?

(I) $f(t) = 6t$ (II) $f(t) = 5t - 20$ (III) $f(t) = 2$
   $g(t) = 3t^2 - 3$ $g(t) = 5t + 15$ $g(t) = 2t$
   $h(t) = 3t - 6$ $h(t) = 10t - 5$ $h(t) = 2t^2$

A) $\boxed{\text{(I) and (III)}}$ B) (III) only   C) (I) only   D) (II) only

To determine linear dependence or independence, we look for linear relations. More specifically, we look for the possibility of having $af + bg + ch = 0$ when some of the scalars a, b, c are non-zero

(I) $0 = 6at + b(3t^2 - 3) + c(3t - 6)$
From the t^2 term, $b = 0$; from the constant term, $b + 2c = 0$, hence $c = 0$; from the t term $2a + c = 0$ so $a = 0$
Thus, the three functions are linearly independent

(II) $0 = a(5t - 20) + b(5t + 15) + c(10t - 5)$
We have to solve the following system of two equations with three unknowns

$$a + b + 2c = 0$$

$$-20a + 15b - 5c = 0$$

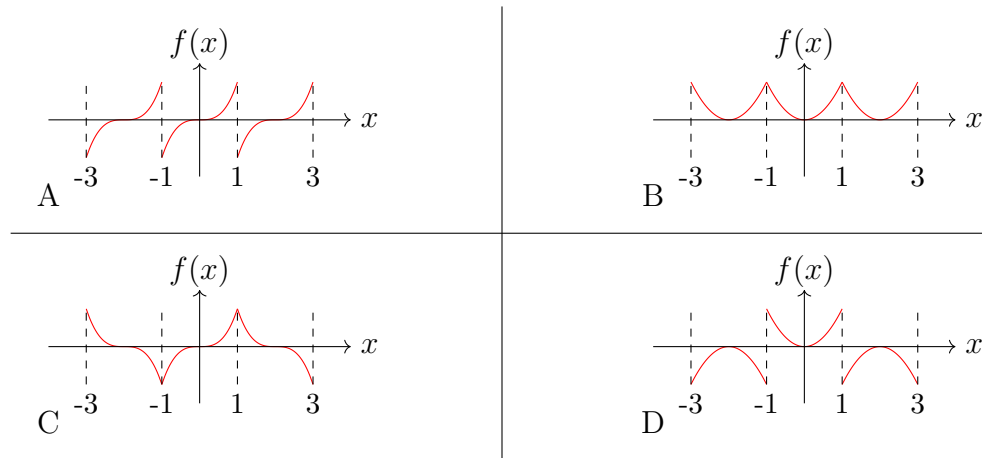Solve in terms of c: $a + b = -2c$, $20a - 15b = -5c$, hence $7a = 3(a + b) + (4a - 3b) = -7c$
Therefore $a = -c$ and $b = -2c - a = -c$
Since c can be non-zero, we show that these three functions are linearly dependent

(III) $0 = 2a + 2bt + 2ct^2$, which gives us $a = b = c = 0$
These three functions are linearly independent

4. The function f is defined on $[0,1)$ by $f(x) = x^2$. Which of the plots below represents the odd 2-periodic extension of f?

ANSWER: (A)

Note (B) and (D) are not odd, and (C) is not 2-periodic. Even functions exhibit the behavior $f(-x) = f(x)$ or are symmetric about the y-axis. Odd functions exhibit the behaviors $f(-x) = -f(x)$ or the graph is symmetric about the origin.

5. The temperature $u(x,t)$ in a bar of length 4 with heat diffusivity $\alpha^2 = \frac{1}{4}$ satisfies the heat equation $\alpha^2 u_{xx} = u_t$. If both ends of the bar are insulated and $u(x,0) = 6$ for $0 \leq x \leq 1$ and 2 for $1 < x \leq 4$, then evaluate the limit as $t \to \infty$ for $u(x,t)$

A) 2

B) 6

C) 3

D) 0

E) 4

The insulated endpoints correspond to zero Neumann boundary conditions
The temperature at each point converges to the constant value $\frac{c_0}{2}$ as t $\to \infty$ (see formula sheet)
The value $\frac{c_0}{2}$ equals the average of the initial temperature $u(x,0)$, which is

$$\frac{1}{4} \int_0^4 u(x,0)dx = \frac{1}{4}(6*1 + 2*3) = \frac{12}{4} = 3$$

6. Which of the following integrals are zero for all $L > 0$?

(I) $$\int_{-L}^{L} \sin(3t)e^{t^2}\,dt$$

(II) $$\int_{-L}^{L} x^3 \cos x\,dx$$

(III) $$\int_{-L}^{L} t\sin(t^4)\,dt$$

(IV) $$\int_{-L}^{L} |x^2 - 5|\sin x\,dx$$

A) (I) only   B) $\boxed{\text{All}}$ C) (IV) only   D) (II) only   E) (III) only

The integral of an odd function on a symmetric interval is equal to 0. An odd function times another odd (or even times even) function is an even function. An odd times an even function is an odd function

7. Functions f, g, h and k are 6-periodic. Their values on $[-3, 3)$ are given below. For which of these functions does the Fourier series converge at $x = 0$ to the value 1?

$$f(x) = \begin{cases} 2 + x & -3 \le x < 0 \\ 1 & x = 0 \\ -2 & 0 < x < 3 \end{cases} \qquad g(x) = \begin{cases} 1 + x & -3 \le x < 0 \\ 4 & x = 0 \\ 2 - x^2 & 0 < x < 3 \end{cases}$$

$$h(x) = \begin{cases} x^2 - 1 & -3 \le x < 0 \\ -1 & x = 0 \\ 3 & 0 \le x < 3 \end{cases} \qquad k(x) = \begin{cases} 3 + x & -3 \le x < 1 \\ 1 & x = 1 \\ x - 8 & 1 < x < 3 \end{cases}$$

A) f

B) $\boxed{h}$

C) None

D) g and k

E) f and h

The Fourier series of a function $\phi(x)$ converges at $x = 0$ to value $\phi(0)$ if ϕ is continuous at $x = 0$, and converges to value $\frac{\phi(0-)+\phi(0+)}{2}$ if ϕ jumps at $x = 0$. Here is the summary of relevant information

f: $f(0-) = 2$, $f(0+) = -2$, jump at $x = 0$, so Fourier series at 0 has value 0

g: $g(0-) = 1$, $g(0+) = 2$, jump at $x = 0$, so Fourier series at 0 has value $\frac{3}{2}$

h: $h(0-) = -1$, $h(0+) = 3$, jump at $x = 0$, so Fourier series at 0 has value 1

k: $k(0) = 3$ and k is continuous at $x = 0$, so Fourier series at 0 has value 3

8. Compute all the eigenvalues and corresponding eigenfunctions for the boundary value problem

$$y'' - \lambda y = 0 \qquad y'(-2) = 0, y(0) = 0$$

If a certain range of the real numbers does not include any eigenvalues, show why there are none in that range

The BVP has eigenvalues

$$\lambda_n = -\frac{(n - \frac{1}{2})^2 \pi^2}{4}$$

with the corresponding eigenfunctions

$$y_n(x) = \sin\left(\frac{(n - \frac{1}{2})\pi x}{2}\right), \ (n = 1,2...)$$

a. There are no positive eigenvalues. Write $\lambda = \mu^2$ (with $\mu > 0$), hence we are solving the diff eq

$$y'' - \mu y = 0$$

The corresponding characteristic equation $r^2 - \mu^2 = 0$ has roots $r = \pm\mu$ hence the general solution is

$$y(x) = C_1 e^{\mu x} + C_2 e^{-\mu x}$$

Then $y(0) = C_1 + C_2 = 0$, hence $C_2 = -C_1$ and $y(x) = C_1(e^{\mu x} - e^{-\mu x})$
Further, $y'(x) = C_1\mu(e^{\mu x} + e^{-\mu x})$ so $y'(-2) = C_1\mu(e^{-2\mu} + e^{2\mu}) = 0$
This leads to $C_1 = 0$, hence $y(x) = 0$

b. Zero is not an eigenvalue either. For $\lambda = 0$ we are solving the DE $y'' = 0$
$y(x) = C_1 x + C_2$. We have $y(0) = C_1 = 0$ and $y'(-2) = C_2 = 0$, hence $y = 0$

c. Finally we look for negative eigenvalues $\lambda = -\mu^2$, with $\mu > 0$. We are solving the diff eq

$$y'' + \mu^2 y = 0$$

The general solution has the form

$$y(x) = C_1 \cos(\mu x) + C_2 \sin(\mu x)$$

From $y(0) = y(0) = C_1$ we conclude that $y(x) = C_2 \sin(\mu x)$ and $y'(x) = C_2\mu \cos(\mu x)$
We have $y'(-2) = C_2\mu \cos(2\mu) = 0$, hence y can be non-zero if and only if $\cos(2\mu) = 0$

This last equality occurs if and only if

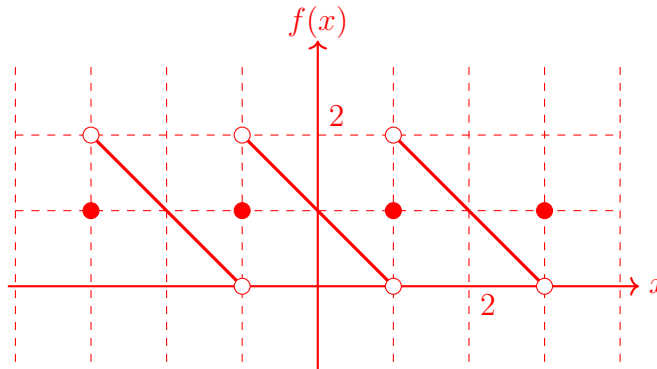$$2\mu = \pi\left(n - \frac{1}{2}\right)$$

for some positive integer n. Thus, we have eigenvalues

$$\lambda_n = -\mu_n^2 = -\frac{(n - \frac{1}{2})^2 \pi^2}{4}$$

with the corresponding eigenfunctions

$$y_n(x) = \sin(\mu_n x) = \sin\left(\frac{(n - \frac{1}{2})\pi x}{2}\right)$$

9. Consider the function $f(x) = 1 - x$ defined on the interval $x \in [-1, 1)$

(a) Sketch the 2-periodic extension of f(x) on the interval $x \in [-3, 3]$

(b) Compute the 2-periodic Fourier series representation of f(x)

Here we have $L = 1$ so the Fourier expansion is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(n\pi x) + b_n \sin(n\pi x) \right), \text{ where } a_n = \int_{-1}^{1} f(x) \sin(n\pi x) dx$$

$$\text{In our case: } a_0 = \int_{-1}^{1} (1 - x) dx = \left[x - \frac{x^2}{2} \right]_{-1}^{1} = 2 \qquad \text{for n} \geq 1$$

$$a_n = \int_{-1}^{1} (1 - x) \cos(n\pi x) dx = \int_{-1}^{1} \cos(n\pi x) dx - \int_{-1}^{1} (1 - x) \cos(n\pi x) dx$$

But $\int_{-1}^{1} (1\text{-x}) \cos(n\pi x) dx = 0$ (we are integrating an odd function on a symmetric interval), and $\int_{-1}^{1} \cos(n\pi x) dx = 0$ (we are integrating an odd function on a symmetric interval)

So, $a_n = 0$

$$b_n = \int_{-1}^{1} (1 - x) \sin(n\pi x) dx = \int_{-1}^{1} \sin(n\pi x) dx - \int_{-1}^{1} (x) \sin(n\pi x) dx$$

We have $\int_{-1}^{1} \sin(n\pi x) dx = 0$ (integral of an odd function)

Solve with Integration by Parts

$$- \int_{-1}^{1} (x) \sin(n\pi x) dx =$$

$$\frac{1}{n\pi} \int_{-1}^{1} x \, d\cos(n\pi x) =$$

$$\frac{1}{n\pi} \left[x \cos(n\pi x) \right]_{-1}^{1} - \frac{1}{n\pi} \int_{-1}^{1} \cos(n\pi x) dx =$$

$$\frac{2\cos(n\pi)}{n\pi} - \frac{1}{n^2\pi^2}\Big[\sin(n\pi x)\Big]_{-1}^{1}$$

$$b_n = \frac{2(-1)^n}{n\pi}$$

This gives us the Fourier expansion

$$\boxed{1 + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n\pi}\sin(n\pi x)}$$

10. Find the general solution of
$$y^{(4)} - 2y'' + y = e^t + t^2$$

(Hint: Use Method of Undetermined Coefficients)

The general solution can be written as

$$y = y_C + Y$$

where y_C is the complementary solution (the general solution of the homogeneous equation), and Y is the particular solution of the original, non-homogeneous equation

First recognize, the ODE is linear with constant coefficients. The characteristic polynomial

$$r^4 - 2r^2 + 1 = (r^2 - 1)^2 = (r - 1)^2(r + 1)^2$$

which has two real roots, ± 1, both of multiplicity 2.
Thus, the complementary solution is

$$y_C = (C_1 t + C_2)e^t + (C_3 t + C_4)e^{-t}$$

Then by undetermined coefficients look for

$$Y = At^2 e^t + Bt^2 + Ct + D$$

where the original guess Ae^t for the exponential part of the particular solution has been multiplied by t^2 to eliminate duplication with y_C, in that part of the guess.

Find Y'' and $Y^{(4)}$

$$Y'' = A(t^2 + 4t + 2)e^t + 2B$$
$$Y^{(4)} = A(t^2 + 8t + 12)e^t$$

Hence the ODE says

$$Y^{(4)} - 2Y'' + Y = 8Ae^t + Bt^2 + Ct + (D - 4B) = e^t + t^2$$

Equating coefficients, we get $A = \frac{1}{8}$, $B = 1$, $C = 0$, $D = 4$. Thus, the general solution is:

$$\boxed{y = (C_1 t + C_2)e^t + (C_3 t + C_4)e^{-t} + \frac{t^2}{8}e^t + t^2 + 4}$$