



Center for Academic Resources in Engineering (CARE) Peer Exam Review Session

Math 241 — Calculus III

Midterm 3 Worksheet Solutions

The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: Apr 26, 8-9:30pm Pieter and Nidhi Session 2: Apr 27, 8-9:30pm Danny and David

Can't make it to a session? Here's our schedule by course:

<https://care.engineering.illinois.edu/tutoring-resources/tutoring-schedule-by-course/>

Solutions will be available on our website after the last review session that we host, as well as posted in the zoom chat 30 minutes prior to the end of the session

Step-by-step login for exam review session:

1. Log into [Queue @ Illinois](#)
2. Click "New Question"
3. Add your NetID and Name
4. Press "Add to Queue"
5. Join the zoom link in the staff message

Please do not log into the zoom call without adding yourself to the queue

Good luck with your exam!

1. Consider the following vector fields $\vec{F}(x, y, z)$. Are they conservative? If so, find a function $f(x, y, z)$ so that $\nabla f = \vec{F}$. If not, justify your response.

Conservative vector field test: a vector field \vec{F} is conservative if the curl is the zero vector.

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left\langle \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right\rangle = \vec{0}$$

a) $\vec{F}(x, y, z) = \langle yz, xz, xy + 2z \rangle$

$$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy + 2z \end{vmatrix} = \langle x - x, y - y, z - z \rangle = \vec{0}$$

The vector field is conservative, therefore, a potential function exists. To find it, we must find the necessary terms from each component (We neglect the constant for now, we'll add it back later).

$$\int F_x \, dx = \int yz \, dx = xyz$$

$$\int F_y \, dy = \int xz \, dy = xyz$$

$$\int F_z \, dz = \int xy + 2z \, dz = xyz + z^2$$

We see that the necessary terms are xyz and z^2 , therefore

The field is conservative and has potential function $f(x, y, z) = xyz + z^2 + C$

b) $\vec{F}(x, y, z) = \langle y + e^x, x - \cos y, 4 + z \rangle$

$$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y + e^x & x - \cos y & 4 + z \end{vmatrix} = \langle 0 - 0, 0 - 0, 1 - 1 \rangle = \vec{0}$$

The vector field is conservative, therefore, a potential function exists. To find it, we must integrate each component (We neglect the constant for now, we'll add it back later).

$$\begin{aligned} \int F_x \, dx &= \int y + e^x \, dx = xy + e^x \\ \int F_y \, dy &= \int x - \cos y \, dy = xy - \sin y \\ \int F_z \, dz &= \int 4 + z \, dz = 4z + \frac{1}{2}z^2 \end{aligned}$$

We see that the necessary terms are $xy, e^x, -\sin y, 4z$, and $\frac{1}{2}z^2$, therefore:

The field is conservative and has potential function $f(x, y, z) = xy + e^x - \sin y + 4z + \frac{1}{2}z^2$

c) $\vec{F}(x, y, z) = \langle y, z^2, x \rangle$

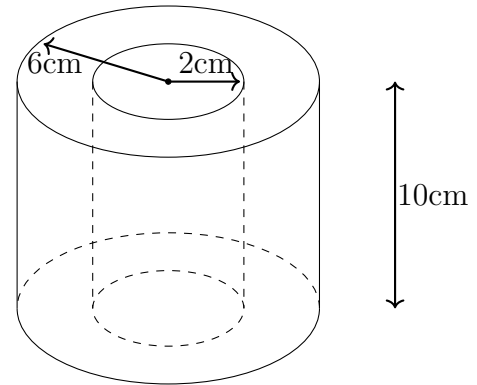
$$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z^2 & x \end{vmatrix} = \langle -2z, 1, -1 \rangle$$

This vector field is not conservative. Therefore, a potential function does not exist.

2. A toilet paper manufacturing company has increased their production. Unfortunately, this production increase has caused a major manufacturing error! As you move towards the center of any one toilet paper roll, the sheets get progressively more dense. The density of a toilet paper roll can be modeled using the following function:

$$\rho(r) = \cos\left(\frac{\pi(r-1)}{6}\right) + 1$$

r is the radial distance away from the center of the roll (inside the center cardboard tube). The whole roll can be modeled as a cylinder with an outer radius of 6, and inner radius of 2 (the cardboard tube radius), and a height of 10.



- (a) Without using a calculator, calculate the mass of the toilet paper roll if the density everywhere was just 1 (leave π in your answer)

The mass of the tube is equal to the density times the volume:

$$m = \rho V = \rho \pi (r_{outer}^2 - r_{inner}^2) h = \boxed{320\pi}$$

- (b) Set up the triple integral to solve for the mass of a toilet paper roll. Neglect the weight of the inner cardboard tube for your calculation.

Cylindrical coordinates are most useful here since the figure is a cylinder. The mass is found by integrating the density with respect to dV (in cylindrical coordinates).

$$\boxed{\int_0^{10} \int_0^{2\pi} \int_2^6 \left(\cos\left(\frac{\pi(r-1)}{6}\right) + 1 \right) r \, dr d\theta dz}$$

- (c) Without using a calculator, solve the integral (leave π in your answer)

$$\begin{aligned} & \int_0^{10} \int_0^{2\pi} \int_2^6 \left(\cos\left(\frac{\pi(r-1)}{6}\right) + 1 \right) r \, dr d\theta dz \\ & (10)(2\pi) \int_2^6 \cos\left(\frac{\pi(r-1)}{6}\right) r \, dr \\ & 20\pi \left[\left(\frac{r^2}{6}\right) \Big|_2^6 + \int_2^6 \cos\left(\frac{\pi(r-1)}{6}\right) r \, dr \right] \end{aligned}$$

Integration by parts

$$u = r, v = \frac{6}{\pi} \sin\left(\frac{\pi}{6}(r-1)\right)$$

$$20\pi \left[18 - 2 + \left(\frac{6}{\pi} \sin\left(\frac{\pi}{6}(r-1)\right)r \right)_2^6 - \int_2^6 \frac{6}{\pi} \sin\left(\frac{\pi}{6}(r-1)\right) dr \right]$$

$$20\pi \left[16 + \frac{36}{\pi} \sin\left(\frac{5\pi}{6}\right) - \frac{12}{\pi} \sin\left(\frac{\pi}{6}\right) - \int_2^6 \frac{6}{\pi} \sin\left(\frac{\pi}{6}(r-1)\right) dr \right]$$

$$20\pi \left[16 + \frac{18}{\pi} - \frac{6}{\pi} + \frac{6^2}{\pi^2} \cos\left(\frac{\pi}{6}(r-1)\right) \right]_2^6$$

$$20\pi \left[16 + \frac{12}{\pi} + \frac{36}{\pi^2} \left(\cos\left(\frac{5\pi}{6}\right) - \cos\left(\frac{\pi}{6}\right) \right) \right]$$

$$20\pi \left[16 + \frac{12}{\pi} + \frac{36}{\pi^2} \left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} \right) \right]$$

$$20\pi \left[16 + \frac{12}{\pi} + \frac{36\sqrt{3}}{\pi^2} \right]$$

$$\boxed{320\pi + 240 - \frac{720\sqrt{3}}{\pi}}$$

3. The vector field $\vec{F} = \langle 2xy + 2x + y^2, 2xy + 2y + x^2 \rangle$ is conservative. Find a potential function f for \vec{F} (a function with $\nabla f = \vec{F}$)

$$\vec{F} = \langle f_x, f_y \rangle = \langle 2xy + 2x + y^2, 2xy + 2y + x^2 \rangle$$

$$\int f_x = yx^2 + x^2 + xy^2 + C(y)$$

$$\int f_y = xy^2 + y^2 + yx^2 + C(x)$$

Looking at all the terms and comparing with \vec{F} , we know that $C(y) = y^2$ and $C(x) = x^2$, therefore the potential function is:

$$\boxed{f(x, y, z) = yx^2 + x^2 + xy^2 + y^2}$$

4. Find the work done by the force field below in moving an object from (1,1) to (2,4) (HINT: Check if the vector field is conservative).

$$\vec{F}(x, y) = \langle 6y^{\frac{3}{2}}, 9x\sqrt{y} \rangle$$

First we need to check if this vector field is conservative

$$\frac{\partial F_x}{\partial y} = 9\sqrt{y} \text{ and } \frac{\partial F_y}{\partial x} = 9\sqrt{y}$$

Since $\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}$ we can say the vector field is conservative

Now we can find a function $f(x, y)$ such that $\nabla f = \vec{F}$

$$\int F_x \, dx = \int 6y^{\frac{3}{2}} \, dx = 6xy^{\frac{3}{2}}$$

$$\int F_y \, dy = \int 9x\sqrt{y} \, dy = 6xy^{\frac{3}{2}}$$

Thus our potential function is

$$f(x, y) = 6xy^{\frac{3}{2}} + C$$

Now that we have a potential function we can use the Fundamental Theorem of Line Integrals to compute the work done in moving from (1,1) to (2,4)

$$\begin{aligned} W &= \int_C \vec{F} \cdot d\vec{r} = f(x_2, y_2) - f(x_1, y_1) = \\ &f(2, 4) - f(1, 1) = (96 + C) - (6 + C) \end{aligned}$$

$W = 90 \text{ (units)}$